A Guide to Tensor Algebra

Here you can find some fundamentals of tensor algebra, intended as a set of rules and operations among tensors. What is a tensor? It is a mathematical object represented by an array of numbers. The structure of these objects depend on the tensorial order and on dimesionality of the space they live in. In the example below we use a 2D space for simplicity. You can extend all what you learn below to the 3D space we live in as an exercise. Let's first take a look at the tensorial order:

0th order tensor: scalar (just a number). Examples: α, β

1st order tensor: vector (a column of numbers). Examples: $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Transposition: $\underline{a}^T = (a_1, a_2)$

2nd order tensor: matrix (an array of numbers). Examples:
(We only consider square matrices)

$$\underset{\sim}{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Identity matrix: $\underset{\sim}{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Transposition: $\underset{\sim}{A}^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$ $\qquad A^T_{ij} = A_{ji} \qquad i,j = 1,2$

Operations between 1st order tensors (**vectors**):

Scalar product or **inner product** (or **single-index contraction**):
- takes 2 vectors as input (1st order) and gives 1 scalar (0th order)

$$\underline{a} \circ \underline{b} = a_1 b_1 + a_2 b_2 = \sum_{i=1}^{2} a_i b_i$$

$$= \underline{b} \circ \underline{a}$$

It has commutative property

Via **Einstein notation** (repeated indices that only appear in one side of the equation indicate a sum): $\underline{a} \circ \underline{b} = a_i b_i$

Via **Matrix multiplication** (see below): $\underline{a} \circ \underline{b} = \underline{a}^T \underline{b} = (a_1 a_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Tensor product or **outer product**:
- takes 2 vectors as input (1st order) and gives 1 matrix (2nd order)

$$\underline{a} \otimes \underline{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

$$(\underline{a} \otimes \underline{b})_{ij} = a_i b_j$$

Vector product or **cross product**:
- takes 2 vectors as input (1st order) and gives 1 vector (1st order)

$$\underline{a} \times \underline{b} = \left(\underset{\approx}{e}^T \underline{a}\right) \underline{b}$$

$$(\underline{a} \times \underline{b})_k = e_{ijk} a_i b_j$$

You can only apply this in 3D (*)

Levi-Civita (permutation) tensor:

$$e_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ or } i=k \text{ or } j=k \\ 1 & \text{if } ijk = 1,2,3 \,;\, 2,3,1 \,;\, 3,1,2 \quad \text{(in order)} \\ -1 & \text{otherwise} \end{cases}$$

Operations between 2nd order tensors (**matrices**):

Matrix product (or **single-index contraction**):

- takes 2 matrices as input (2nd order) and gives 1 matrix (2nd order)

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \underset{\sim}{A}\,\underset{\sim}{B}$$

$$(\underset{\sim}{A}\,\underset{\sim}{B})_{ij} = A_{ik} B_{kj} = A_{i1} B_{1j} + A_{i2} B_{2j}$$

$$(\underset{\sim}{A}\,\underset{\sim}{B})_{11} = A_{11} B_{11} + A_{12} B_{21}$$

Scalar product or **inner product** (or **double-index contraction**):

- takes 2 matrices as input (2nd order) and gives 1 scalar (0th order)

$$\underset{\sim}{A} \circ \underset{\sim}{B} = \underset{\sim}{A} : \underset{\sim}{B}$$

$$\underset{\sim}{A} : \underset{\sim}{B} = A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{21} B_{21} + A_{22} B_{22}$$

$$= \underset{\sim}{B} : \underset{\sim}{A}$$

It has commutative property

Trace: an invariant of the matrix (i.e. does not change with the reference system [see below])

- takes 1 matrix as input (2nd order) and gives 1 scalar (0th order)

$$tr(\underset{\sim}{A}) = A_{ii} = A_{ij}\delta_{ij} = \underset{\sim}{A} : \underset{\sim}{I} = A_{11}\underbrace{\cancel{\delta_{11}}}^{1} + \cancel{A_{12}\,\delta_{12}}^{\,0} + \cancel{A_{21}\,\delta_{21}}^{\,0} + A_{22}\underbrace{\cancel{\delta_{22}}}^{1}$$

$$= A_{11} + A_{22}$$

Kronecker Delta : $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\qquad \delta_{ij} = I_{ij}$

Interesting property:

$$tr(\underset{\sim}{A}) = tr(\underset{\sim}{A}^T) \qquad tr(\underset{\sim}{A}^T) = \underset{\sim}{A}^T : \underset{\sim}{I} = A^T_{ij}\,\underset{=\delta_{ji}}{\delta_{ij}} = A_{ji}\,\delta_{ji} = A_{ii}$$

Determinant: another invariant of the matrix

- takes 1 matrix as input (2nd order) and gives 1 scalar (0th order)

$$det(\underset{\sim}{A}) = A_{11} A_{22} - A_{12} A_{21}$$

In 3D you can use the Levi-Civita permutation tensor: $det(\underset{\sim}{A}) = e_{ijk} A_{1i} A_{2j} A_{3k}$
or (**)

(*) Another way to do the cross product: $\underline{a} \times \underline{b} = det\begin{pmatrix} \hat{\underline{x}}_1 & \hat{\underline{x}}_2 & \hat{\underline{x}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

Minor matrix: $\underset{\sim}{M}$ is minor matrix of $\underset{\sim}{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

if $M_{11} = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = A_{22} A_{33} - A_{23} A_{32}$ Determinant

$M_{12} = \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} = A_{21} A_{33} - A_{23} A_{31}$ ...

Cofactor matrix: $\underset{\sim}{C}$ is cofactor matrix of $\underset{\sim}{A}$ if $C_{ij} = (-1)^{i+j} M_{ij}$

Minor Matrix

$$\underset{\sim}{C} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}$$

Cofactor Matrix

Adjugate matrix: $\underset{\sim}{Ad}$ is adjugate matrix of $\underset{\sim}{A}$ if $\underset{\sim}{Ad} = \underset{\sim}{C}^{T}$

Inverse matrix: $\underset{\sim}{A}^{-1}$ is inverse matrix of $\underset{\sim}{A}$ if $\underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{A}\,\underset{\sim}{A}^{-1} = \underset{\sim}{I}$

$$\underset{\sim}{A}^{-1} = \frac{1}{\det(\underset{\sim}{A})}\underset{\sim}{Ad}$$

Determinant (simple way): $\det(\underset{\sim}{A}) = \underline{a}_i \cdot \underline{c}_i$ $\quad$ $\underline{a}_i, \underline{c}_i$: i-th row or column of $\underset{\sim}{A}$ and $\underset{\sim}{C}$

Cofactors

More properties:

$$\underset{\sim}{A} : \underset{\sim}{B} = \underset{\sim}{B} : \underset{\sim}{A} = B_{ij} A_{ij} = A_{ij} B_{ij}$$

$$\underset{\sim}{A} : \underset{\sim}{B} = \underbrace{tr(\underset{\sim}{A}\,\underset{\sim}{B}^{T})}_{= (\underset{\sim}{A}\,\underset{\sim}{B}^{T}) : \underset{\sim}{I} = A_{ij} B^{T}_{jk} \delta_{ik} = A_{ij} B_{ij}} = tr(\underset{\sim}{A}^{T}\underset{\sim}{B}) = tr(\underset{\sim}{B}\,\underset{\sim}{A}^{T}) = tr(\underset{\sim}{B}^{T}\underset{\sim}{A})$$

Tensorial functions and fuctionals: $\alpha = \alpha(\underset{\sim}{A})$ $\quad$ α is function of $\underset{\sim}{A}$

$$\frac{\partial \alpha}{\partial \underset{\sim}{A}} : \quad \left(\frac{\partial \alpha}{\partial \underset{\sim}{A}}\right)_{ij} = \frac{\partial \alpha}{\partial A_{ij}}$$

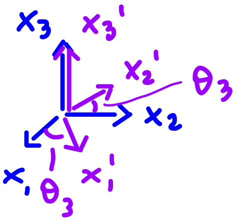
$$\underset{\sim}{A} = \underset{\sim}{A}(\alpha) \qquad \frac{\partial \underset{\sim}{A}}{\partial \alpha} : \quad \left[\frac{\partial \underset{\sim}{A}}{\partial \alpha}\right]_{ij} = \frac{\partial A_{ij}}{\partial \alpha}$$

$$\underset{\sim}{A} = \underset{\sim}{A}(\underset{\sim}{B}) \qquad \frac{\partial \underset{\sim}{A}}{\partial \underset{\sim}{B}} : \quad \left(\frac{\partial \underset{\sim}{A}}{\partial \underset{\sim}{B}}\right)_{ijhk} = \frac{\partial A_{ij}}{\partial B_{hk}}$$

Matrix-vector multiplication: $\underline{b} = \underset{\sim}{A}\,\underline{a} \qquad b_i = A_{ij}\, a_j$

$$b_1 = A_{11} a_1 + A_{12} a_2 \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} A_{11} a_1 + A_{12} a_2 \\ A_{21} a_1 + A_{22} a_2 \end{pmatrix}$$
$$b_2 = A_{21} a_1 + A_{22} a_2$$

Rotation Matrix: Changing the reference system



$$\underset{\sim}{R}_3(\theta_3) = \begin{array}{c} \begin{array}{ccc} \cos\theta_3 & \sin\theta_3 & \end{array} \\ \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

rotation around x_3 ($x_3 = x_3'$)

$$\underset{\sim}{R}_3^T(\theta) = \underset{\sim}{R}_3(-\theta) = \underset{\sim}{R}_3^{-1}(\theta) = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

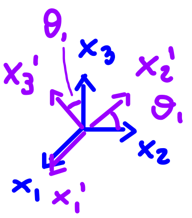
Proof:

Unit vectors $\underline{\hat{x}}_i$:

$$\underline{\hat{x}}_1' = c\,\underline{\hat{x}}_1 + s\,\underline{\hat{x}}_2 + 0\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_2' = -s\,\underline{\hat{x}}_1 + c\,\underline{\hat{x}}_2 + 0\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_3' = 0\,\underline{\hat{x}}_1 + 0\,\underline{\hat{x}}_2 + 1\,\underline{\hat{x}}_3$$

$$\underline{a} = a_i'\,\underline{\hat{x}}_i' = a_1' c\,\underline{\hat{x}}_1 + a_1' s\,\underline{\hat{x}}_2 - a_2' s\,\underline{\hat{x}}_1 + a_2' c\,\underline{\hat{x}}_2 + a_3'\,\underline{\hat{x}}_3$$

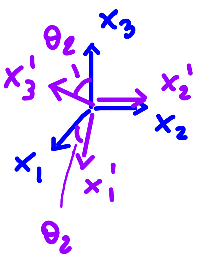
$$= \begin{pmatrix} a_1' c - a_2' s \\ a_1' s + a_2' c \\ a_3' \end{pmatrix} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad : \quad \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$



$$\underset{\sim}{R}_1 = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \\ \begin{array}{cc} \sin\theta_1 & \cos\theta_1 \end{array} \end{array}$$

rotation around x_1

Proof:
$$\underline{\hat{x}}_1' = 1\,\underline{\hat{x}}_1 + 0\,\underline{\hat{x}}_2 + 0\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_2' = 0\,\underline{\hat{x}}_1 + c\,\underline{\hat{x}}_2 + s\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_3' = 0\,\underline{\hat{x}}_1 - s\,\underline{\hat{x}}_2 + c\,\underline{\hat{x}}_3 \qquad \ldots$$



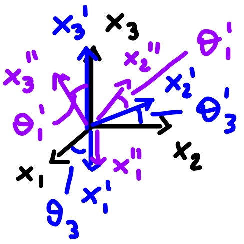
$$\underset{\sim}{R}_2 = \begin{array}{c} \begin{array}{cc} \cos\theta_2 & \sin\theta_2 \end{array} \\ \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \end{array}$$

rotation around x_2

Proof:
$$\underline{\hat{x}}_1' = c\,\underline{\hat{x}}_1 + 0\,\underline{\hat{x}}_2 - s\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_2' = 0\,\underline{\hat{x}}_1 + 1\,\underline{\hat{x}}_2 + 0\,\underline{\hat{x}}_3$$
$$\underline{\hat{x}}_3' = s\,\underline{\hat{x}}_1 + 0\,\underline{\hat{x}}_2 + c\,\underline{\hat{x}}_3$$

$\longleftarrow$ Inverse "chronological" order (*)

Composition: $\underset{\sim}{R} = \underset{\sim}{R}_3 \, \underset{\sim}{R}_2 \, \underset{\sim}{R}_1$

Example: 

$$(x_1, x_2, x_3) \xrightarrow{\underset{\sim}{R}_3} (x_1', x_2', x_3') \xrightarrow{\underset{\sim}{R}_{1'}} (x_1'', x_2'', x_3'')$$

$$\underline{a}' = \underset{\sim}{R}_3 \, \underline{a} \qquad \underline{a}'' = \underset{\sim}{R}_{1'} \, \underline{a}' = \underbrace{\underset{\sim}{R}_{1'} \, \underset{\sim}{R}_3}_{\underset{\sim}{R}} \, \underline{a}$$

$\longleftarrow$ (*)

$$\underset{\sim}{R} = \underset{\sim}{R}_{1'} \, \underset{\sim}{R}_3$$

Vector : $\underline{a}' = \underset{\sim}{R} \, \underline{a}$

Matrix : $\underset{\sim}{A}' = \underset{\sim}{R} \, \underset{\sim}{A} \, \underset{\sim}{R}^T$

Proof : Take $\underline{b} = \underset{\sim}{A} \underline{a}$; Since $\underline{b}' = \underset{\sim}{R} \, \underline{b}$ and $\underline{a} = \underset{\sim}{R}^T \underline{a}'$

$$\underset{\sim}{A}' \underline{a}' = \underline{b}' = \underset{\sim}{R} \, \underline{b} = \underset{\sim}{R} \, \underset{\sim}{A} \, \underline{a} = \underset{\sim}{R} \, \underset{\sim}{A} \, \underset{\sim}{R}^T \underline{a}' \qquad \underset{\sim}{A}' = \underset{\sim}{R} \, \underset{\sim}{A} \, \underset{\sim}{R}^T$$

- If you change the reference system, say from x1,x2,x3 to x1′,x2′,x3′, the components of the vectors and the matrices change (assume different values)

Eigenvalues and Eigenvectors: Matrix diagonalization

A square symmetric matrix can always be diagonalized, i.e. You can find a specific coordinate system x1′,x2′,x3′ for which you can write your matrix as a diagonal one : $\underset{\sim}{A}' = \underset{\sim}{A}_P = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$

Principal Matrix

$$\underset{\sim}{A} \, \underline{x}_i' = \alpha_i \, \underline{x}_i' \quad : \quad (\underset{\sim}{A} - \alpha_i \underset{\sim}{I}) \, \underline{x}_i' = 0$$

Eigenvalue Eigenvector

$$\underline{x}_i' \neq \underline{0} \quad \text{only if} \quad \det(\underset{\sim}{A} - \alpha_i \underset{\sim}{I}) = 0$$

$$\det \begin{pmatrix} A_{11} - \alpha_i & A_{12} \\ A_{21} & A_{22} - \alpha_i \end{pmatrix} = (A_{11} - \alpha_i)(A_{22} - \alpha_i) - A_{12} A_{21} = 0$$

$$\alpha_i = \tfrac{1}{2}(A_{11} + A_{22}) \pm \tfrac{1}{2} \sqrt{(A_{11} - A_{22})^2 + 4 A_{12} A_{21}}$$

2 values : α_1 & α_2

$(\underset{\sim}{A} - \alpha_i \underset{\sim}{I}) \, \underline{x}_i' = \underline{0}$: $\underline{x}_i' = \ldots$ 2 ortogonal directions

- The eigenvalues of the matrix are also called **principal values**. They include the maximum and the minimum values each component of the matrix can ever achieve in any reference system. This is why the maximum principal stress is the maximum tensile stress experienced by the material in the analyzed point.